

Subgroup theorem for valuated groups and the CSA property

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Abstract

A valuated group with normal forms is a group with an integer-valued length function satisfying some Lyndon's axioms [Lyn63] and an additional axiom considered by Hurley [Hur80]. We prove a subgroup theorem for valuated groups with normal forms analogous to Grushko-Neumann's theorem. We study also the CSA property in such groups.

1 Introduction

Let (Λ, \leq) be an (totally) ordered abelian group, and G a group with a length function $\ell : G \rightarrow \Lambda$. For $x, y \in G$, we let $c(x, y) = \frac{1}{2}(\ell(x) + \ell(y) - \ell(xy^{-1}))$. We notice that we may have $c(x, y) \notin \Lambda$, but we may assume that we are working in the divisible ordered abelian closure of Λ (see [Chi01] for more details).

We say that ℓ is a *Lyndon length function*, if it satisfies the following axioms considered by Lyndon [Lyn63]:

- \mathcal{A}_1 . $\ell(1) = 0$,
- \mathcal{A}_2 . for all $x \in G$, $\ell(x^{-1}) = \ell(x)$,
- \mathcal{A}_3 . for all $x, y, z \in G$, $c(x, y) \geq \min\{c(x, z), c(z, y)\}$,

and in that case (G, ℓ) is called a Λ -*valuated group*. If Λ is \mathbb{Z} with the usual ordering, we call (G, ℓ) a *valuated group*. We shall use the notation ℓ for length functions unless otherwise indicated.

In [Chi76], Chiswell showed that a valuated group (G, ℓ) , assuming that $c(x, y)$ is always an integer, acts on a tree T in such a way that $\ell(g)$ is the tree distance between p and gp for some suitable vertex p of T . Conversely, if T is a tree and p is vertex of T , and G is a group acting on T , then (G, ℓ) is a valuated group, $\ell(g)$ being the tree distance between p and gp . Hence, the subject of valuated groups fits in the theory of groups acting on trees.

We are concerned in this paper with a restricted class of valuated groups. We are interested in valuated groups G , which satisfy the following additional axiom considered by Hurley [Hur80]:

- \mathcal{A}_4 . G is generated by the set $\{x \in G \mid \ell(x) \leq 1\}$.

A valued group G satisfying \mathcal{A}_4 is called a *valuated group with normal forms*. Free groups, free products with amalgamation and HNN-extensions are the typical examples of valued groups with normal forms. Proposition 3.3 below, shows that every element of a valued groups with normal forms has normal forms, having several properties comparable to those of free product with amalgamations and HNN-extensions.

It is very pleasant to work directly in the class of valued groups with normal forms for several raisons. For instance, because this class contains free groups, free product with amalgamation and HNN-extensions, and in several times the same proof works for all such groups as it depends generally on some properties of normal forms. Therefore, this provides a unifying framework in which we can study both free products with amalgamation and HNN-extensions.

Lyndon [Lyn63] has introduced integer-length functions on groups, satisfying some axioms including $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$, to axiomatize the argument of Nielsen's proof of the subgroup theorem for free groups. He has proved a splitting theorem (Theorem 3.1 below), which gives a new proof of the Nielsen subgroup theorem, the Kurosh subgroup theorem, and which gives more information about the restriction of the natural length on subgroups of free products. Hurley [Hur80] has studied groups with normal forms (called NFS-groups), and has shown that such groups can be obtained by considering Lyndon's axioms and the additional axiom \mathcal{A}_4 above.

By introducing a new axiom (\mathcal{A}_5^* below), we prove a subgroup theorem for valued groups analogous to Grushko-Neumann's theorem (Theorem 4.1). This gives at least an uniform statement and an uniform proof of theorems about subgroups in free groups, free product with amalgamation and HNN-extensions. We prove a splitting theorem for valued groups satisfying a special axiom (\mathcal{A}_0^* below), and we study centralizers in valued groups with normal forms.

We will be also interested in this paper, with the CSA property. A subgroup H of a group G is *conjugately separated* in G , or *malnormal* in G , if $H \cap H^x = 1$ for every $x \in G \setminus H$. A *CSA-group* ("Conjugately Separated Abelian") is a group in which every maximal abelian subgroup is malnormal. It is known that a CSA-group with an *involution*, i.e. an element of order 2, must be abelian. Following the notation of [MR96], we denote by *CSA*-group* any CSA-group without involutions. The class of CSA-groups contains free groups, and more generally torsion-free hyperbolic groups [Gro87], groups acting freely on Λ -trees [Bas91, Chi01], and limit groups [GS93, Rem89].

In [GKM95, JOH04], conditions on free product with amalgamation and HNN-extension to be a CSA*-groups are given. We generalize in this paper that results to valued groups with normal forms.

If (G, ℓ) is Λ -valuated group, it follows from the axioms, and we leave the proof to the reader, the following properties (which will be used freely without explicit reference):

- (i) $c(x, y) = c(y, x)$ and $\ell(x) \geq 0$, for all $x, y \in G$.
- (ii) $|\ell(x) - \ell(y)| \leq \ell(xy) \leq \ell(x) + \ell(y)$, for all $x, y, z \in G$, where $|\cdot|$ is the absolute value in Λ . In particular, if $\ell(y) = 0$ then $\ell(xy) = \ell(x)$.

We define $B = \{x \in G \mid \ell(x) = 0\}$. Then it follows from \mathcal{A}_1 - \mathcal{A}_2 and (ii) that B is a subgroup of G .

We consider also the following type of length functions. Let G be a group generated by some set S . Then one defines a length function ℓ_S over G , called *word length*, as follows. For every $g \in G$, we let $\ell_S(g)$ to be the smallest length of words w over $S^{\pm 1}$ such that $g = w$. Then ℓ_S takes its values in \mathbb{N} , and one can check easily that ℓ_S satisfies \mathcal{A}_1 , \mathcal{A}_2 and $\ell_S(xy) \leq \ell_S(x) + \ell_S(y)$, for all $x, y \in G$.

The paper is organized as follows. In the next section we give some examples of valuated groups. Section 3 is devoted to preliminaries on valuated groups. In Section 4, we show a subgroup theorem for valuated groups with normal forms. We prove in Section 5, a splitting theorem for valuated groups satisfying axiom \mathcal{A}_0^* ; which is a weak version of axiom \mathcal{A}_0 . Section 6 concerns conjugacy in valuated groups with normal forms, and we study in Section 7 centralizers in such groups. We prove in Section 8, a theorem which gives sufficient conditions for a valuated group to be a CSA*-group.

We denote by $X^\#$ the set of nontrivial elements of X , and g^h denotes $h^{-1}gh$.

2 Examples & other axioms

We give now some examples of Λ -valuated groups.

(1) Typical examples.

- *Free groups.* The group (F, ℓ_X) , where F is a free group with basis X and ℓ_X is the word length function, is a valuated group satisfying the following axiom (introduced by Lyndon):

$$\mathcal{A}_0. \quad \ell(x^2) > \ell(x), \text{ for every } x \neq 1.$$

Conversely, Lyndon [Lyn63] has shown that a valuated group satisfying \mathcal{A}_0 is also a free group. Remark that in this case we have $B = \{1\}$.

- *Free products.* Let $G = G_1 * G_2$. Put $S = (G_1 \cup G_2) \setminus \{1\}$. Then (G, ℓ_S) is a valuated group which satisfies the following axiom (introduced by Lyndon):

$$\mathcal{A}_5. \quad c(x, y) + c(x^{-1}, y^{-1}) > \ell(x) = \ell(y) \Rightarrow x = y$$

Of course $\ell_S(g)$ is the length of normal forms of g . Remark also that in this case we have $B = \{1\}$.

- *Amalgamated free product.* Let $G = G_1 *_A G_2$. Let $\ell(g)$ be the length of the normal form of g if $g \notin A$ and $\ell(g) = 0$ if $g \in A$. Then (G, ℓ) is a valuated group satisfying the Chiswell's axioms [Chi81]:

$$C1'. \quad \text{If } \ell(x) \text{ is even and } \ell(x) \neq 0, \text{ then } \ell(x^2) > \ell(x),$$

$$C2. \quad \text{for no } x \text{ is } \ell(x^2) = 1 + \ell(x).$$

Conversely, Chiswell [Chi81] has shown that a valuated group with normal forms (G, ℓ) satisfying the above axioms is a free product of a family $\{G_i \mid i \in I\}$

with a subgroup A amalgamated, such that ℓ is the natural length function relative to this decomposition. Note that we have $B = A$.

- *HNN-extensions.* Let $G^* = \langle G, t | A^t = B \rangle$ be an HNN-extension. Let $\ell(x)$ be the number of occurrences of t^\pm in the normal form of x . Then (G, ℓ) is a valuated group with normal forms, with in this case $B = G$. Remark that we have $\ell(t) = 1$ and $\ell(t^2) = 2$, unlike the the case of amalgamated free products where we have $\ell(x^2) \leq 1$ for every x such that $\ell(x) \leq 1$.

(2) Model of the universal theory of non-abelian free groups.

Let \mathcal{M} be a model of the universal theory of nonabelian free groups. Then \mathcal{M} embeds in an ultrapower *F of F_2 which can be equipped with a Lyndon function ${}^*\ell$ taking its values in an ultrapower ${}^*\mathbb{Z}$ of \mathbb{Z} and satisfying \mathcal{A}_0 . We get that $(\mathcal{M}, {}^*\ell|_{\mathcal{M}})$ is a ${}^*\mathbb{Z}$ -valuated group satisfying \mathcal{A}_0 .

This viewpoint was used by Chiswell and Remeslennikov [CR00], to give a new proof of a theorem of Appel and Lorents, about the solutions of equations with one variable in free groups.

(3) Groupes acting on Λ -tree.

In [Chi76, Chi01] Chiswell considered valuated groups and has shown that such groups can be obtained from their action on a suitable tree. More generally he has shown that if (G, ℓ) is Λ -valuated group, then G acts by isometry on Λ -tree and if (X, d) is a Λ -tree and if G is a group acting by isometry over X , by defining for $x \in X$ $\ell(g) = d(x, gx)$, then (G, ℓ) is a Λ -valuated group.

(4) Some linear groups.

Let (K, v) be a fields with a discrete valuation v . Then, by a result of Serre [Ser80], $\mathrm{GL}_2(K)$ acts on tree. Therefore, it can be equipped with a Lyndon length function. Chiswell [Chi77] has given the explicit form of that function, and has shown that the corresponding tree is the same as that constructed by Serre. If we take \mathbb{Q} with the p -adic valuation v_p , we get $\mathrm{GL}_2(\mathbb{Q})$ as a valuated group.

3 Preliminary

3.1 Valuated groups

We define some interesting subsets and relations of a Λ -valuated group (G, ℓ) . We let

$$N_G = \{g \in G \mid \ell(g^2) \leq \ell(g)\},$$

and we denote it N if there is no risque of ambiguity. We let \equiv to be the following relation, defined in N , by:

$$x \equiv y \text{ if and only if } \ell(x^{-1}y) \leq \ell(x) = \ell(y).$$

This relation is du to Lyndon [Lyn63]. One can check easily that \equiv is an equivalence relation on N . Indeed, obviously \equiv is reflexive and symmetric. Let $x, y, z \in N$ such that $x \equiv y, y \equiv z$. Then $\ell(x) = \ell(y) = \ell(z)$ and

$$c(x, y) = \frac{1}{2}(\ell(x) + \ell(y) - \ell(x^{-1}y)) \geq \frac{1}{2}\ell(x),$$

$$c(y, z) = \frac{1}{2}(\ell(y) + \ell(z) - \ell(y^{-1}z)) \geq \frac{1}{2}\ell(y),$$

and by axiom \mathcal{A}_3 we have $c(x, z) \geq \frac{1}{2}\ell(y)$. Hence $\ell(x^{-1}z) \leq \ell(x) = \ell(y) = \ell(z)$, thus $x \equiv z$.

We denote by $N^*(x)$ the equivalence class of x under \equiv . Then we see that $N^*(1) = B$. We let $N(x) = N^*(x) \cup \{1\}$.

Let U be a subset of G . Let (u_1, \dots, u_n) be a sequence in $U^{\pm 1}$. We say that (u_1, \dots, u_n) is *pseudo-reduced* if:

- (i) $u_i u_{i+1} \neq 1$, $u_i \neq 1$,
- (ii) if $u_i, u_{i+1} \in U^{\pm 1} \cap N$, then $u_i \neq u_{i+1}$.

We need in the sequel the following theorem of Lyndon, which can be extracted from its results in [Lyn63].

Theorem 3.1 [Lyn63] *Let (G, ℓ) be a valuated group satisfying*

$$\mathcal{A}_1^*. \ell(x) = 0 \Rightarrow x = 1,$$

$$\mathcal{A}_5. c(x, y) + c(x^{-1}, y^{-1}) > \ell(x) = \ell(y) \Rightarrow x = y.$$

Then

- (1) *For every $x \in N$, the set $N(x)$ is a subgroup of G .*
- (2) *There exists a generating set U of G such that:*
- (i) *for every pseudo-reduced sequence (u_1, \dots, u_n) of $U^{\pm 1}$ we have:*

$$\ell(u_1 \cdots u_n) = \sum_{i=1}^n \ell(u_i) - 2 \sum_{i=1}^{n-1} c(u_i, u_{i+1}^{-1}),$$

(ii) *we have $G = F *_i G_i$, where F is a free group having a basis $X \subseteq U$ and $G_i = N(x)$ for some $x \in U \cap N$.* \square

We end this section with the following lemma, needed in the sequel, and which generalizes the one proved by Lyndon [Lyn63] in case of valuated groups satisfying $\mathcal{A}_1^*, \mathcal{A}_5$. The proof is an exact copy of the one in [OH06], and it is left to the reader.

Lemma 3.2 *Let (G, ℓ) be a Λ -valuated group. Let (g_1, \dots, g_n) , $n \geq 2$, be a sequence in G satisfying:*

$$c(g_{i-1}, g_i^{-1}) + c(g_i, g_{i+1}^{-1}) < \ell(g_i), \quad 1 < i < n.$$

$$\text{Then } \ell(g_1 \cdots g_n) = \sum_{i=1}^n \ell(g_i) - 2 \sum_{i=1}^{n-1} c(g_i, g_{i+1}^{-1}). \quad \square$$

3.2 Valuated groups with normal forms

Hurley [Hur81], after studying some groups with normal forms, have noticed that the typical examples cited earlier and the groups that he have studied satisfy the axiom

$$\mathcal{A}_4. G \text{ is generated by the set } \{x \in G \mid \ell(x) \leq 1\}$$

As it was said in the introduction, we call a valuated group (G, ℓ) , a *valuated group with normal forms* if (G, ℓ) satisfies \mathcal{A}_4 .

We let $S = \{x \in G \mid \ell(x) \leq 1\}$. A sequence (s_1, s_2, \dots, s_n) of S is said to be *S-reduced* if $s_i \cdot s_{i+1} \notin S$ for every $1 \leq i \leq n-1$.

The following proposition shows that every element in a valuated group with normal forms has normal forms. His proof can be extracted from [Hur80], but for completeness we provide a proof.

Proposition 3.3 [Hur80] *Let (G, ℓ) be a valuated group with normal forms. Then:*

- (1) *If (s_1, \dots, s_n) , with $n \geq 2$, is a S-reduced sequence, then*

$$\ell(s_1 \cdots s_n) = \ell_S(s_1 \cdots s_n) = n.$$

- (2) *For every $g \in G \setminus B$, there exists a S-reduced sequence (s_1, \dots, s_n) such that $g = s_1 \cdots s_n$ and $\ell(g) = \ell_S(g) = n$. Thus if (s'_1, \dots, s'_m) is a S-reduced sequence such that $g = s'_1 \cdots s'_m$, then $m = n$.*

Proof

- (1) Since (s_1, \dots, s_n) is S-reduced, we get $c(s_i, s_{i+1}^{-1}) = 0$ for $1 \leq i \leq n-1$. Therefore, by Lemma 3.2, $\ell(s_1 \cdots s_n) = \sum_{i=1}^{i=n} \ell(s_i)$.

- (2) The existence of the S-reduced sequence follows from the fact that G is generated by S and the rest follows from (1). \square

Definition 3.4 *A normal form of g is a S-reduced sequence (s_1, \dots, s_n) such that $g = s_1 \cdots s_n$ (If $g \in B$, a normal form of g is g). If $x = s_1 \cdots s_n$, where (s_1, \dots, s_n) is S-reduced, we say that x is written in normal form.*

Proposition 3.5 *Let (G, ℓ) be a valuated group with normal forms. Let $x, y \in G$, $x = s_1 \cdots s_n$ and $y = t_1 \cdots t_m$ in normal forms. If $\ell(s_n t_1) = 2$, then the sequence $(s_1, \dots, s_n, t_1, \dots, t_m)$ is a normal form of xy . If $\ell(s_n t_1) = 1$, then $(s_1, \dots, (s_n t_1), \dots, t_m)$ is a normal form of xy .*

Proof

Clearly if $\ell(s_n t_1) = 2$, then the sequence $(s_1, \dots, s_n, t_1, \dots, t_m)$ is S-reduced, and therefore it is a normal form of xy .

Suppose that $\ell(s_n t_1) = 1$. Then it is sufficient to show that $\ell(s_{n-1} s_n t_1) = \ell(s_n t_1 t_2) = 2$.

Suppose towards a contradiction that $\ell(s_{n-1} s_n t_1) < 2$. Then

$$c(s_{n-1}, (s_n t_1)^{-1}) \geq \frac{1}{2},$$

and since $c(s_{n-1}, s_n^{-1}) = 0$, we find by axiom \mathcal{A}_3 that

$$c(s_{n-1}, s_n t) = c(s_n, t_1^{-1}) = 0,$$

and thus $\ell(s_n) + \ell(s_n t_1) - \ell(t_1) = 0$, which is clearly a contradiction.

Suppose now towards a contradiction that $\ell(s_n t_1 t_2) < 2$. Then

$$c(s_{n-1} s_n, t_1^{-1}) \geq \frac{1}{2},$$

and since $c(t_1, t_2^{-1}) = 0$, we find by axiom \mathcal{A}_3 that

$$c(t_1, s_n t_1) = c(t_1, t_2^{-1}) = 0,$$

and thus $\ell(t_1) + \ell(s_n t_1) - \ell(s_n) = 0$, which is clearly a contradiction. \square

Definition 3.6 Let (G, ℓ) be a valuated group with normal forms.

(1) An element $g \in G$ is said *cyclically reduced, abbreviated c.r.*, if $g \in S$ or $g = s_1 \cdots s_n$ in normal form ($n \geq 2$) and $\ell(s_n s_1) = 2$. This definition does not depend on the particular choice of the normal form of g , as we see that if $\ell(g) \geq 2$ then g is c.r. if and only if $\ell(g^2) = 2\ell(g)$.

(2) An element $g \in G$ is said *weakly cyclically reduced, abbreviated w.c.r.*, if $g \in S$ or $g = s_1 \cdots s_n$ in normal form ($n \geq 2$) and $\ell(s_n s_1) \neq 0$. As before, this definition does not depend on the particular choice of the normal form of g , as we see that if $\ell(g) \geq 2$ then g is w.c.r. if and only if $\ell(g^2) \geq 2\ell(g) - 1$.

Lemma 3.7 Let (G, ℓ) be a valuated group with normal forms. Then every element g of G is conjugate to a c.r. element. Furthermore if $g \in N$ then, there exist $x, y \in G$ such that x is c.r., $g = y^{-1}xy$, $\ell(g) = 2\ell(y) + \ell(x)$ and $x \in N \cap S$.

Proof

The proof is by induction on $\ell(g)$. The result is clear when $\ell(g) \leq 1$.

Let $g = s_1 \cdots s_n$ in normal form, with $n \geq 2$. If $\ell(s_1 s_2) = 2$, then g is c.r. and there is no thing to prove. If $\ell(s_n s_1) = 1$, then $g = s_n^{-1}(s_n s_1 s_2 \cdots s_{n-1})s_n$. Now the sequence $(s_{n-1}, s_n s_1)$ is S -reduced and therefore $s_n s_1 s_2 \cdots s_{n-1}$ is a c.r. element. If $\ell(s_n s_1) = 0$, then $s_1 = s_n^{-1}h$ for some $h \in B$ and thus $g = s_n^{-1}(hs_2 \cdots s_{n-1})s_n$. But $\ell(hs_2 \cdots s_{n-1}) = n-1$, and by induction it is conjugate to a c.r. element, and the same thing holds also for g .

Suppose now that $g \in N$. The proof is by induction on $\ell(g)$. If $\ell(g) \leq 1$ the result is clear. Let $g = s_1 \cdots s_n$ in normal form with $n \geq 2$. Since $g \in N$ we have $\ell(g^2) \leq \ell(g)$ and therefore we get $\ell(s_n s_1) = 0$; as otherwise $\ell(g^2) = 2\ell(g) - 1 > \ell(g)$. Then $s_n = hs_1^{-1}$ for some $h \in B$.

If $n = 2$, then $g = s_1 s_2 = s_1 h s_1^{-1}$, and thus we have the desired conclusion.

For $n \geq 3$, we have $g = s_1(s_2 \cdots s_{n-1}h)s_1^{-1}$ for some $h \in B$. Let $g' = s_2 \cdots s_{n-1}h$. Since $\ell(g) = 2 + \ell(g')$, the conclusion follows by induction if we show that $g' \in N$. But a simple count with normal forms shows that $\ell(g^2) = \ell(g'^2) + 2 \leq \ell(g) = 2 + \ell(g')$, and thus $g' \in N$ as desired. \square

4 Subgroup Theorem for valuated groups

The subject of this section is to prove the following theorem.

Theorem 4.1 (Grushko-Neumann version for valuated groups) *Let (G, ℓ) be a valuated group with normal forms. Let K be a subgroup of G such that all conjugates of K intersect B trivially. Then $K = F *_i G_i$, where F is a free group and $G_i = K \cap N(x_i)^{a_i}$ for some $a_i \in G$ and $x_i \in S$ such that $\ell(x_i^2) \leq 1$.*

We introduce the following axiom:

$$\mathcal{A}_5^*. \text{ If } c(x, y) + c(x^{-1}, y^{-1}) > \ell(x) = \ell(y), \text{ then } xy^{-1} \in B^G,$$

where B^G is the set of all conjugates of B .

We show first the following.

Proposition 4.2 *A valuated group with normal forms satisfies the axiom \mathcal{A}_5^* .*

It follows in particular, that a free product with amalgamation or an HNN-extension satisfies \mathcal{A}_5^* .

Proof

Let $x, y \in G$ such that

$$c(x, y) + c(x^{-1}, y^{-1}) > \ell(x) = \ell(y).$$

Then, after simplifications, we find

$$(1) \quad \ell(xy^{-1}) + \ell(x^{-1}y) < 2\ell(x).$$

Clearly the case $\ell(x) = \ell(y) = 0$ is impossible.

Let us treat the case $\ell(x) = \ell(y) = 1$. Then, by (1), $\ell(xy^{-1}) + \ell(x^{-1}y) < 2$. Hence $\ell(xy^{-1}) = 0$ or $\ell(x^{-1}y) = 0$.

If $\ell(xy^{-1}) = 0$, then $xy^{-1} \in B$ and we are done. If $\ell(x^{-1}y) = 0$, then $x^{-1}y = b$ for some $b \in B$, and thus $xy^{-1} = xb^{-1}x^{-1} \in B^G$. This ends the proof in the case $\ell(x) = \ell(y) = 1$.

Suppose now that $\ell(x) = \ell(y) = n \geq 2$. By (1) we have

$$(2) \quad \ell(xy^{-1}) + \ell(x^{-1}y) < 2n,$$

and in particular $\ell(xy^{-1}) < 2n$.

Let $x = s_1 \cdots s_n$ and $y = t_1 \cdots t_n$ in normal forms. Since

$$\ell(xy^{-1}) < 2n = \ell(x) + \ell(y),$$

we get $\ell(s_n t_n^{-1}) \leq 1$.

If $\ell(s_n t_n^{-1}) = 1$, then $\ell(xy^{-1}) = 2n - 1$, and by (2) we get $\ell(x^{-1}y) = 0$. Then $x^{-1}y = b$ for some $b \in B$, and thus $xy^{-1} = xb^{-1}x^{-1} \in B^G$.

So we suppose that $\ell(s_n t_n^{-1}) = 0$.

Let i be the smallest natural number such that $\ell(s_i \cdots s_n t_n^{-1} \cdots t_i^{-1}) = 0$. Then we can write:

$$x = ab, \quad \ell(x) = \ell(a) + \ell(b), \quad \text{where } a = s_1 \cdots s_{i-1}, \quad b = s_i \cdots s_n,$$

$$y = \alpha\beta, \quad \ell(y) = \ell(\alpha) + \ell(\beta), \quad \text{where } \alpha = t_1 \cdots t_{i-1}, \quad \beta = t_i \cdots t_n.$$

We notice that $\ell(a) = \ell(\alpha)$ and $\ell(b) = \ell(\beta)$.

Let $w = a^{-1}\alpha\beta b^{-1}$.

Claim 1. We have $\ell(b^{-1}w) = \ell(b^{-1}) + \ell(w)$.

Proof. Since $c(x, y) + c(x^{-1}, y^{-1}) > \ell(x) = \ell(y)$ and $\ell(y) - c(x, y) = c(y^{-1}, xy^{-1})$ we find

$$c(x^{-1}, y^{-1}) > c(y^{-1}, xy^{-1}),$$

and by using axiom \mathcal{A}_3 , we get

$$(3) \quad c(y^{-1}, xy^{-1}) = c(x^{-1}, xy^{-1}).$$

A simplification of the expression appearing in (3) gives:

$$\ell(x^{-1}yx^{-1}) = \ell(x), \quad \text{and thus}$$

$$(4) \quad \ell(b^{-1}wa^{-1}) = \ell(b^{-1}a^{-1}\alpha\beta b^{-1}a^{-1}) = \ell(ab).$$

From $\ell(\beta b^{-1}) = 0$, we get

$$(5) \quad \ell(b^{-1}w) = \ell(b^{-1}a^{-1}\alpha(\beta b^{-1})) = \ell(b^{-1}a^{-1}\alpha),$$

$$(6) \quad \ell(w) = \ell(a^{-1}\alpha(\beta b^{-1})) = \ell(a^{-1}\alpha).$$

Since $\ell(ab) = \ell(a) + \ell(b)$ we have

$$\ell(b^{-1}a^{-1}) = \ell(b^{-1}) + \ell(a^{-1}),$$

and since $\ell(a) = \ell(\alpha)$, we get

$$\ell(b^{-1}a^{-1}\alpha) = \ell(b^{-1}) + \ell(a^{-1}\alpha).$$

Therefore, using (5) and (6), we get

$$\ell(b^{-1}w) = \ell(b^{-1}) + \ell(w),$$

as claimed. □

Claim 2. We have $\ell(a^{-1}\alpha) = 0$.

Proof. We treat the following two cases.

Case 1. $\ell(b^{-1}wa^{-1}) = \ell(b^{-1}w) + \ell(a^{-1})$.

Using (4) and Claim 1 we get

$$\ell(a) + \ell(b) = \ell(b^{-1}wa^{-1}) = \ell(b^{-1}w) + \ell(a^{-1}) = \ell(b^{-1}) + \ell(w) + \ell(a^{-1}).$$

It follows that $\ell(w) = 0$ and therefore, by (6), $\ell(a^{-1}\alpha) = 0$ as desired.

Case 2. $\ell(b^{-1}wa^{-1}) < \ell(b^{-1}w) + \ell(a^{-1})$.

Then $c(b^{-1}w, a) > 0$ and since $c(b^{-1}, a) = 0$, by using A_2 we find

$$c(b^{-1}, b^{-1}w) = 0.$$

A simplification of the above expression gives

$$\ell(\beta^{-1}\alpha^{-1}ab) = \ell(b) + \ell(b^{-1}w).$$

Therefore, using (6) and Claim 1 we get

$$\ell(x^{-1}y) = \ell(y^{-1}x) = \ell(\beta^{-1}\alpha^{-1}a.b) = 2\ell(b) + \ell(a^{-1}\alpha).$$

But, counting with normal forms we get

$$(8) \quad \begin{aligned} \ell(xy^{-1}) &= \ell(a(b\beta^{-1})\alpha) = \ell(a) + \ell(\alpha) \\ \text{or } \ell(xy^{-1}) &= \ell(a(b\beta^{-1})\alpha) = \ell(a) + \ell(\alpha) - 1. \end{aligned}$$

and thus

$$\begin{aligned} \ell(xy^{-1}) + \ell(x^{-1}y) &= 2\ell(b) + \ell(a^{-1}\alpha) + \ell(a) + \ell(\alpha), \\ \text{or } \ell(xy^{-1}) + \ell(x^{-1}y) &= 2\ell(b) + \ell(a^{-1}\alpha) + \ell(a) + \ell(\alpha) - 1. \end{aligned}$$

Since $\ell(b) = \ell(\beta)$, we get

$$\begin{aligned} \ell(xy^{-1}) + \ell(x^{-1}y) &= 2n + \ell(a^{-1}\alpha), \\ \text{or } \ell(xy^{-1}) + \ell(x^{-1}y) &= 2n + \ell(a^{-1}\alpha) - 1. \end{aligned}$$

By (2), the first case is impossible. Again by (2), we find $\ell(a^{-1}\alpha) - 1 < 0$ and finally $\ell(a^{-1}\alpha) = 0$ as desired. This ends the proof of the claim. \square

Therefore $\ell(a^{-1}yx^{-1}a) = \ell(a^{-1}\alpha\beta b^{-1}) = 0$, and thus $a^{-1}yx^{-1}a = b$ for some $b \in B$. Hence $yx^{-1} \in B^G$ as desired. \square

Proof of Theorem 4.1.

We let K equipped with the induced length function. Since G satisfies \mathcal{A}_5^* and K satisfies $gKg^{-1} \cap B = \{1\}$ for any $g \in G$, we find that K satisfies \mathcal{A}_5 and \mathcal{A}_1^* . We notice that for any $x \in K$, $N_K(x) = K \cap N_G(x)$. Therefore by Theorem 3.1 and since \equiv is an equivalence relation, we have

$$K = F *_i G_i, \text{ where } F \text{ is free and } G_i = K \cap N_G(y_i), \text{ with } y_i \in N_G \cap K.$$

In what follows we denote by $N(x)$ (resp. N) the set $N_G(x)$ (resp. N_G).

We are going to show that $G_i = K \cap N(x_i)^{a_i}$ for some $a_i \in K$ and $x_i \in S \cap N$.

If $y_i \in S \cap N$, then there is no thing to prove. So we suppose that $y_i \notin S \cap N$.

By Lemma 3.7 we have

$$y_i = s_1 \cdots s_n h s_n^{-1} \cdots s_1^{-1},$$

$$\text{with } \ell(s_j) = 1, \ell(y_i) = 2n + \ell(h), \ell(h) \leq 1, h \in N.$$

Since $(s_1 \cdots s_n)^{-1} G_i (s_1 \cdots s_n) \cap B = \{1\}$, we have $\ell(h) = 1$.

Let us show that $K \cap s_1 \cdots s_n N(h) s_n^{-1} \cdots s_1^{-1} = K \cap N(y_i)$.

We first show that $K \cap s_1 \cdots s_n N(h) s_n^{-1} \cdots s_1^{-1} \subseteq K \cap N(y_i)$.

Let $\beta \in N(h)$ and $y = s_1 \cdots s_n \beta s_n^{-1} \cdots s_1^{-1} \in K$. Since $\beta \equiv h$ and $\ell(h) = 1$ we have $\ell(\beta) = 1$. Thus $\ell(y) \leq 2n + 1$. Let us show that $\ell(y) = 2n + 1$. Suppose towards a contradiction that $\ell(y) < 2n + 1$. Then, using properties of normal forms, we have

$$\ell(s_n \beta) \leq 1 \quad \text{or} \quad \ell(\beta s_n^{-1}) \leq 1$$

and since $\beta \equiv h$, we must have

$$\ell(s_n h) \leq 1 \quad \text{or} \quad \ell(h s_n^{-1}) \leq 1.$$

Therefore $\ell(y_i) < 2n + \ell(h)$, contradiction with $\ell(y_i) = 2n + \ell(h)$.

Thus $\ell(y) = 2n + 1 = \ell(y_i)$ as desired. But we also have

$$\ell(y_i y) = \ell(s_1 \cdots s_n h \beta s_n^{-1} \cdots s_1^{-1}) \leq \ell(y).$$

Therefore $y_i \equiv y$, hence $y \in K \cap N(x_i)$ as desired.

We show now that $K \cap s_1 \cdots s_n N(h) s_n^{-1} \cdots s_1^{-1} \supseteq K \cap N(y_i)$.

Let $y \in K \cap N(y_i)$. Then by Lemma 3.7, there exists β such that

$$y = t_1 \cdots t_n \beta t_n^{-1} \cdots t_1^{-1}.$$

Since $y_i \equiv y$, we have $\ell(\beta) = 1$ and $\ell(y_i y) \leq \ell(y_i)$.

Therefore

$$\ell(h s_n^{-1} \cdots s_1^{-1} t_1 \cdots t_n \beta) \leq 1$$

and thus $\ell(s_n^{-1} \cdots s_1^{-1} t_1 \cdots t_n) = 0$. Therefore $t_1 \cdots t_n = s_1 \cdots s_n \gamma$ where $\gamma \in B$.

Thus

$$y = s_1 \cdots s_n \gamma \beta \gamma^{-1} s_n^{-1} \cdots s_1^{-1}.$$

Since $\gamma \beta \gamma^{-1} \in N$ and $\ell(h s_n^{-1} \cdots s_1^{-1} t_1 \cdots t_n \beta) = \ell(h \gamma \beta) = \ell(h \gamma \beta \gamma^{-1}) \leq 1$, we find $h \equiv \gamma \beta \gamma^{-1}$.

Therefore $y \in K \cap s_1 \cdots s_n N(h) s_n^{-1} \cdots s_1^{-1}$ and this ends the proof of the theorem. \square

5 A special splitting theorem

If B is a group and f is an automorphism of B , we denote by $B(t, f)$ the HNN-extension $\langle B, t | f(b) = b^t \rangle$. The subject of this section is to prove the next theorem. This theorem is needed in the proof of Theorem 7.1.

Theorem 5.1 *Let (G, ℓ) be a valuated group satisfying the following axiom:*

$$\mathcal{A}_0^*. \text{ If } \ell(x) \neq 0, \text{ then } \ell(x^2) > \ell(x).$$

*Then either $G = B$ or there exists a sequence of automorphisms $(f_i | i \in \lambda)$ of B , and a sequence of elements $(t_i | i \in \lambda)$ of G such that $G = *_B B(t_i, f_i)$.*

We will use the following lemma of Hoare.

Lemma 5.2 [Hoa76] *Let (G, ℓ) be a valuated group. Then*

- (1) *If $\ell(x, y) + \ell(x^{-1}, y^{-1}) \geq \ell(x) = \ell(y)$, then $xy^{-1} \in N$.*
- (2) *Let (u_0, \dots, u_n) be a sequence of G and let*

$$a_i = u_0 \cdots u_{i-1}, i = 0, \dots, n, \text{ and } c_i = u_{i+1} \cdots u_n, \quad i = 1, \dots, n-1.$$

*If $\ell(a_{i+1}) \geq \ell(a_i), \ell(u_i)$ and $\ell(c_{i-1}) \geq \ell(c_i)$ for every $i = 1, \dots, n-1$,
and $\ell(u_0 \cdots u_n) < \ell(u_1 \cdots u_n)$,
then $\ell(a_{i+1}) = \ell(a_i), \ell(c_{i-1}) = \ell(c_i)$ for every $i = 1, \dots, n-1$. \square*

Definition 5.3 *Let (G, ℓ) be a valuated group. Let $\mathcal{U} \subseteq G$. We say that \mathcal{U} is weakly reduced, if for every sequence (u_0, \dots, u_n) of $\mathcal{U} \cup \mathcal{U}^{-1}$ which satisfies $u_i u_{i+1} \neq 1$ and $\ell(u_i) \neq 0$, then $\ell(u_0 \cdots u_n) \geq \ell(u_1 \cdots u_n)$.*

The proof of the following lemma follows the general line of the proof of [Hoa76, Theorem (page 190)].

Lemma 5.4 *Let (G, ℓ) be a valuated group satisfying the axiom \mathcal{A}_0^* . Let \mathcal{U} be a weakly reduced subset of G and let $g \in G$ such that $g \notin \mathcal{U}^{\pm 1}$. Let $\mathcal{U}_* = \mathcal{U} \cup \{g\}$. If \mathcal{U}_* is not weakly reduced and if $\ell(g) \geq \ell(u)$ for every $u \in \mathcal{U}$, then there exists a Nielsen transformation ϕ of \mathcal{U}_* such that $\phi(u) = u$ for every $u \in \mathcal{U}$ and $\ell(\phi(g)) < \ell(g)$.*

Proof

Let (u_0, u_1, \dots, u_n) be a sequence of $\mathcal{U}_*^{\pm 1}$ of minimal length for which \mathcal{U}_* is not weakly reduced. Let a_i and c_i be the sequences as defined in Lemma 5.2. Since n is minimal we have $\ell(a_{i+1}) \geq \ell(a_i), \ell(u_i)$ and $\ell(c_{i-1}) \geq \ell(c_i)$ for every $i = 1, \dots, n-1$, and therefore by Lemma 5.2, $\ell(c_{i-1}) = \ell(c_i)$ for every $i = 1, \dots, n-1$.

Since \mathcal{U} is weakly reduced, there exists i such that $u_i = g^{\pm 1}$. If i is unique then the transformation defined by

$$\phi(g) = u_0 \cdots u_n, \quad \phi(u) = u \text{ for every } u \in \mathcal{U},$$

is a Nielsen transformation and we have

$$\ell(g) \geq \ell(u_n) = \ell(c_{n-1}) = \ell(c_0) > \ell(u_0 \cdots u_n) = \ell(\phi(g)).$$

So it is sufficient to show that the set $\{i | u_i = g^{\pm 1}\}$ is reduced to a one element. Suppose towards a contradiction that there exists $0 \leq i < j \leq n$ such that $u_i = u_j = g^{\pm 1}$ or $u_i = g$ and $u_j = g^{-1}$ or $u_i = g^{-1}$ and $u_j = g$.

We have

$$c(c_{k-1}, c_k) = \frac{1}{2}(\ell(c_{k-1}) + \ell(c_k) - \ell(u_k)), \text{ for every } k = 1, \dots, n-1,$$

and since $\ell(c_k) = \ell(c_0)$ and $\ell(u_i) \geq \ell(u_k)$ we find

$$(1) \quad c(c_{k-1}, c_k) \geq (\ell(c_0) - \frac{1}{2}\ell(u_i)), \quad k = 0, \dots, n-1.$$

We also have

$$c(c_k^{-1}, u_k) = \frac{1}{2}(\ell(c_k) + \ell(u_k) - \ell(c_{k-1})) = \frac{1}{2}\ell(u_k), \quad k = 1, \dots, n-1.$$

$$c(c_0^{-1}, u_0) = \frac{1}{2}(\ell(c_0) + \ell(u_0) - \ell(u_0 c_0)),$$

and since $\ell(u_0 c_0) < \ell(c_0)$ we find $c(c_0^{-1}, u_0) \geq \frac{1}{2}\ell(u_0)$. Therefore

$$(2) \quad c(c_k^{-1}, u_k) \geq \frac{1}{2}\ell(u_k), \quad k = 0, \dots, n-1.$$

Now we treat the following three cases.

Case 1. $u_i = u_j = g^{\pm 1}$, $0 \leq i < j \leq n-1$.

By applying (2) we have

$$c(c_i^{-1}, u_i) \geq \frac{1}{2}\ell(u_i), \text{ and } c(c_j^{-1}, u_j) \geq \frac{1}{2}\ell(u_j).$$

But since $u_i = u_j$, by axiom \mathcal{A}_4 , we find $c(c_i^{-1}, c_j^{-1}) \geq \frac{1}{2}\ell(u_i)$, and thus using (1) we get

$$c(c_i, c_j) + c(c_i^{-1}, c_j^{-1}) \geq \ell(c_i) = \ell(c_j) = \ell(c_0).$$

Therefore by Lemma 5.2, we find $c_j c_j^{-1} \in N$ and by axiom \mathcal{A}_0^* we get $\ell(c_i c_j^{-1}) = 0$, and thus $\ell(u_{i+1} \cdots u_j) = 0$.

Suppose that $\ell(u_{i+2} \cdots u_j) = 0$. Then $\ell(u_{i+1}) = \ell(u_{i+1} \cdots u_j) = 0$, a contradiction with $\ell(u_{i+1}) = 0$.

Therefore $\ell(u_{i+2} \cdots u_j) \neq 0$, and thus $\ell(u_{i+1} \cdots u_j) = 0 < \ell(u_{i+1} \cdots u_j) \neq 0$. Hence the sequence (u_{i+1}, \dots, u_j) contradicts the minimality of the sequence (u_0, \dots, u_n) .

Case 2. $u_i = u_j = g^{\pm 1}$, $0 \leq i < j = n$.

Since $u_i = u_n = c_{n-1}$, then

$$c(c_i, c_{n-1}) \geq \ell(c_0) - \frac{1}{2}\ell(u_i) = \ell(c_{n-1}) - \frac{1}{2}\ell(u_n) = \frac{1}{2}\ell(c_{n-1}),$$

and

$$c(c_i^{-1}, c_{n-1}) = c(c_i^{-1}, u_i) = \frac{1}{2}\ell(u_i) = \frac{1}{2}\ell(c_{n-1}).$$

Applying axiom \mathcal{A}_4 we find

$$c(c_i, c_i^{-1}) = \frac{1}{2}\ell(c_{n-1}) = \frac{1}{2}\ell(c_i).$$

Therefore $\ell(c_i^2) \leq \ell(c_i)$, and thus by axiom \mathcal{A}_0^* , $\ell(c_i) = 0$. Hence $\ell(u_{i+1} \cdots u_{j-1}) = 0$. The conclusion follows as in the previous case.

Case 3. $u_i = u_j^{-1}$.

Then $0 \leq i < j - 1 \leq n - 1$. By writting $c_n = 1$ we find

$$c(c_{j-1}^{-1}, u_j) = \frac{1}{2}(\ell(c_{j-1}) + \ell(u_j) - \ell(c_j)) \geq \frac{1}{2}\ell(u_j),$$

and

$$c(c_i^{-1}, u_i) \geq \frac{1}{2}\ell(u_i).$$

By axiom \mathcal{A}_4 we get $c(c_i^{-1}, c_i^{-1}) \geq \frac{1}{2}\ell(u_i)$. Thus, by (1) we find

$$c(c_i, c_{j-1}) + c(c_i^{-1}, c_{j-1}^{-1}) \geq \ell(c_i) = \ell(c_{j-1}) = \ell(c_0),$$

and thus $\ell(u_{i+1} \cdots u_{j-1}) = 0$. The conclusion follows also as in the previous cases. \square

The following lemma is a simple application of Zermilo theorem and the proof is left to the reader.

Lemma 5.5 *Let G be a group equipped with an integer length function $\ell : G \rightarrow \mathbb{N}$. Then there exists a well ordering \prec of G such that for every $x, y \in G$, if $\ell(x) < \ell(y)$, then $x \prec y$.* \square

Proof of Theorem 5.1.

We may assume that $B \neq G$. Let \prec be a well ordering of G satisfying the conclusion of Lemma 5.5. For every $g \in G$, we let G_g to be the subgroup generated by the set $\{x \in G \mid x \prec g\}$. We let

$$U = \{g \in G \mid g \notin G_g\}, \quad U' = \{g \in U \mid \ell(g) \neq 0\}.$$

Claim 1. U generates G .

Proof. Let H be the subgroup generated by U and suppose towards a contradiction that $G \neq H$. Let a be the smallest element of G which is not in H . Then every element $b \prec a$ is in H and thus $a \notin G_a$. Hence $a \in U$, a contradiction. \square

Claim 2. U is weakly reduced.

Proof. For every $x \in U$, we let $U_x = \{y \in U \mid y \prec x\}$. We show that if U_x is weakly reduced then $U_x \cup \{x\}$ is weakly reduced.

Suppose towards a contradiction that for some $x \in U$, U_x is weakly reduced and $U_x \cup \{x\}$ is not weakly reduced. We see that $x \in U_x^{\pm 1}$ and for every $y \in U_x$, $\ell(x) \geq \ell(y)$. By Lemma 5.2, there exists a Nielsen transformation ϕ such that $\phi(u) = u$ for any $u \in U_x$ and $\ell(\phi(x)) < \ell(x)$. Therefore $\phi(x) \in G_x$ and since $U_x \subseteq G_x$, we find $x \in G_x$; which is a contradiction with $x \in U$.

Thus if U_x is weakly reduced then $U_x \cup \{x\}$ is weakly reduced as required. Hence by induction on the well ordering \prec , U is weakly reduced. \square

Claim 3. B is a normal subgroup of G .

Proof. Let $b \in B$ and $x \in G$. We have

$$c(bx, x^{-1}bx) = \frac{1}{2}(\ell(bx) + \ell(x^{-1}bx) - \ell(x)) = \frac{1}{2}\ell(x^{-1}bx),$$

and similarly,

$$c(bx, (x^{-1}bx)^{-1}) = \frac{1}{2}\ell(x^{-1}bx),$$

and by using axiom \mathcal{A}_4 we find

$$c(x^{-1}bx, (x^{-1}bx)^{-1}) \geq \frac{1}{2}\ell(x^{-1}bx).$$

Therefore $\ell(x^{-1}bx) \geq \ell((x^{-1}bx)^2)$ and thus by axiom \mathcal{A}_0^* we get $x^{-1}bx \in B$. \square

Claim 4. Let (h_0, \dots, h_n) be a sequence of B and (u_0, \dots, u_n) be a sequence of U' . If for every $0 \leq i \leq n-1$, $u_i u_{i+1} \neq 1$, then $\ell(h_0 u_0 \cdots h_n u_n) \geq \ell(h_1 u_1 \cdots h_n u_n)$.

Proof. By Claim 3, B is a normal subgroup of G and thus

$$\ell(h_0 u_0 \cdots h_n u_n u_n^{-1} u_{n-1}^{-1} \cdots u_0^{-1}) = 0,$$

and therefore,

$$\ell(h_0 u_0 \cdots h_n u_n) = \ell(u_0 \cdots u_n),$$

Similarly we have $\ell(h_1 u_1 \cdots h_n u_n) = \ell(u_1 \cdots u_n)$.

Now the conclusion follows from the fact that U is weakly reduced. \square

For each $u \in U'$, we let $F(u) = \langle B, u \rangle$.

Claim 5. Let $f_u(x) = x^u$, restricted to B . Then $F(u) = B(u, f_u)$.

Proof. It is sufficient to show that if (h_0, \dots, h_n) is a sequence of B and (u_1, \dots, u_n) is a sequence of $\{u, u^{-1}\}$ such that for every $0 \leq i \leq n-1$, $u_i u_{i+1} \neq 1$, then $h_0 u_0 \cdots h_n u_n \neq 1$. But this is a consequence of Claim 4. \square

Claim 6. G is the free product of $B(u, f_u)$, for $u \in U'$, amalgamating B .

Proof. First of all, we need to show that $F(u) \cap F(v) = B$, for $u, v \in U'$, $u \neq v$. Let $g \in F(u) \cap F(v)$ and suppose that $\ell(g) > 0$. Let $g = h_0 u^{\varepsilon_1} \cdots h_{n-1} u^{\varepsilon_n} h_n$ (resp. $g = h'_0 v^{\varepsilon'_1} \cdots h'_{n-1} v^{\varepsilon'_m} h'_m$) in normal form relatively to the HNN-structure of $F(u)$ (resp. $F(v)$). Then

$$h_0 u^{\varepsilon_1} \cdots h_{n-1} u^{\varepsilon_n} h_n h'_m{}^{-1} v^{-\varepsilon'_m} \cdots v^{-\varepsilon'_1} h'_0{}^{-1} = 1,$$

which is a contradiction with Claim 4. Thus $F(u) \cap F(v) = B$ as desired.

Now the same argument, by Claim 4, shows that G is the free product of $B(u, f_u)$, for $u \in U'$, amalgamating B . \square

6 Conjugacy in valuated groups with normal forms

The subject of this section is to prove the next theorem which is a generalization of [JOH04, Theorem 6.1] to valuated groups with normal forms.

Theorem 6.1 *Let (G, ℓ) be a valuated group with normal forms. Let x, y, z in G such that $\ell(y) \geq 2$, $y^x = z$. Suppose that y is c.r. or w.c.r as well as z . Then there exist a, b in G , and n and m in \mathbb{Z} , with $n \geq 1$, such that $y = (ab)^n$, $z = (ba)^n$, $x = a(ba)^m$, and such that:*

- (i) *ab and ba are in reduced form whenever y and z are c.r.*
- (ii) *ab and ba are in semi-reduced form whenever y and z are w.c.r.*
- (iii) *ab is in reduced form and ba is in semi-reduced form whenever y is w.c.r. and z is c.r.*

The rest of this section is devoted to the proof of Theorem 6.1, so we adopt all the assumptions and the notation of the statement of that theorem for the rest of this section. The proof is in fact analogous to the one of [JOH04, Theorem 6.1] by taking care here of the existence of elements of length 0. We shall reproduce the proof with the necessary modifications.

We first treat the case $\ell(x) = 0$. By putting $b = x^{-1}, a = yx$ we find $y = ab, z = ba$ and $x = b^{-1} = a(ba)^{-1}$. Since $\ell(x) = 0$ we see that ab and ba are in reduced form.

Now, we shall treat the three cases (i)–(iii) separately.

Case (i): y and z c.r.

Let $x = x_1 \cdots x_p$, $y = y_1 \cdots y_n$, and $z = z_1 \cdots z_m$ in normal forms. We are going to prove the theorem by induction on $p = \ell(x)$.

We first treat the case $p = 1$. Then $x^{-1}y_1 \cdots y_n x = z$. Since y and z are c.r. and $\ell(y) \geq 2$, we have $\ell(x^{-1}y_1) = 0$ or $\ell(y_n x) = 0$. If $\ell(x^{-1}y_1) = 0$, then, $x = y_1 \gamma$ for some $\gamma \in B$. Since y is c.r., $\ell(y_n y_1) = 2$, thus $\ell(y_n x) = \ell(y_n y_1 \gamma) = 2$ and $z = (\gamma^{-1}y_2 \cdots y_n) \cdot x$ is in reduced form. By putting $b = \gamma^{-1}y_2 \cdots y_n$ and $a = x$, we have $y = ab, z = ba$ and $x = a$.

Now, if $\ell(y_n x) = 0$, then $x = y_n^{-1} \gamma$ for some $\gamma \in B$. Since y is c.r., $\ell(y_n y_1) = 2$, thus $\ell(x^{-1}y_1) = \ell(\gamma^{-1}y_n y_1) = 2$, then, $z = x^{-1} \cdot (y_1 \cdots y_{n-1})$ is in reduced

form. By putting $a = y_1 \cdots y_{n-1} \gamma$ and $b = x^{-1}$, we have $y = ab$, $z = ba$, and $x = b^{-1} = a(ba)^{-1}$.

We pass from p to $p+1$ as follows. We have

$$x_{p+1}^{-1} \cdots x_1^{-1} y_1 \cdots y_n x_1 \cdots x_{p+1} = z.$$

Since y and z are c.r. and $\ell(y) \geq 2$, we have $\ell(x_1^{-1} y_1) = 0$ or $\ell(y_n x_1) = 0$. We first treat the case $\ell(x_1^{-1} y_1) = 0$.

Case (1): $\ell(x_1^{-1} y_1) = 0$.

Then $x_1^{-1} y_1 = \gamma$ for some $\gamma \in B$. Then we have

$$x_{p+1}^{-1} \cdots x_2^{-1} \gamma y_2 \cdots y_n y_1 \gamma^{-1} x_2 \cdots x_{p+1} = z.$$

Put $x' = x_2 \cdots x_{p+1}$ and $y' = \gamma y_2 \cdots y_n y_1 \gamma^{-1}$. Then y' is c.r. and $\ell(y') \geq 2$. By induction there exist $a_1, b_1, \alpha \geq 1$ and β such that $y' = (a_1 b_1)^\alpha$, $z = (b_1 a_1)^\alpha$, and $x' = a_1 (b_1 a_1)^\beta$.

Subcase (1-a): $\ell(a_1) = 0$ or $\ell(b_1) = 0$.

Then $y' = C^\alpha$, $z = (\delta^{-1} C \delta)^\alpha$ and $x' = \delta(\delta^{-1} C \delta)^s$ for some $s \in \mathbb{Z}$, where $C = a_1 b_1$, and $\delta = a_1$ whenever $\ell(a_1) = 0$ and $\delta = b_1^{-1}$ whenever $\ell(b_1) = 0$.

Since y' is c.r., C is c.r. Thus we can write $C = C'(y_1 \gamma^{-1})$ in reduced form for some C' , and $(y_1 \gamma^{-1}) C'$ is also in reduced form. Put $a = y_1 \gamma^{-1} \delta$ and $b = \delta^{-1} C'$. Then

$$\begin{aligned} y &= y_1 \gamma^{-1} y' \gamma y_1^{-1} = y_1 \gamma^{-1} (C' y_1 \gamma^{-1})^\alpha \gamma y_1^{-1} = \\ &= y_1 \gamma^{-1} (C' y_1 \gamma^{-1})^{\alpha-1} C' = (y_1 \gamma^{-1} C')^\alpha = (ab)^\alpha, \\ z &= (\delta^{-1} C \delta)^\alpha = (\delta^{-1} C' y_1 \gamma^{-1} \delta)^\alpha = (ba)^\alpha \\ x &= x_1 x' = y_1 \gamma^{-1} \delta (\delta^{-1} C' y_1 \gamma^{-1} \delta)^s = a(ba)^s. \end{aligned}$$

Subcase (1-b): $\ell(a_1) \neq 0$ or $\ell(b_1) \neq 0$.

Since $\ell(b_1) \neq 0$ we can write $b_1 = B'(y_1 \gamma^{-1})$ in reduced form for some B' . Put $a = y_1 \gamma^{-1} a_1$ and $b = B'$. Then

$$\begin{aligned} y &= y_1 \gamma^{-1} y' \gamma y_1^{-1} = y_1 \gamma^{-1} (a_1 B' y_1 \gamma^{-1})^\alpha y_1 \gamma^{-1} = \\ &= y_1 \gamma^{-1} (a_1 B' y_1 \gamma^{-1})^{\alpha-1} a_1 B' = (y_1 \gamma^{-1} a_1 B')^\alpha = (ab)^\alpha, \end{aligned}$$

$$z = (b_1 b_1)^\alpha = (B' y_1 \gamma^{-1} a_1)^\alpha = (ba)^\alpha,$$

where ab and ba are in reduced forms, and

$$x = x_1 x' = y_1 \gamma^{-1} a_1 (b_1 a_1)^\beta = y_1 \gamma^{-1} a_1 (B' y_1 \gamma^{-1} a_1)^\beta = a(ba)^\beta.$$

Case (2): $\ell(y_n x_1) = 0$.

By taking inverses we get

$$x_{p+1}^{-1} \cdots x_1^{-1} y_n^{-1} \cdots y_1^{-1} x_1 x_2 \cdots x_{p+1} = z^{-1}.$$

Therefore, by case (1), there exist a_1, b_1, α and β such that $y^{-1} = (a_1 b_1)^\alpha$, $z^{-1} = (b_1 a_1)^\alpha$ and $x = a_1 (b_1 a_1)^\beta$. Now, by taking $a = b_1^{-1}$ and $b = a_1^{-1}$, we have $y = (ab)^\alpha$, $z = (ba)^\alpha$ and

$$x = a_1 (b_1 a_1)^\beta = b^{-1} (a^{-1} b^{-1})^\beta = a a^{-1} b^{-1} (a^{-1} b^{-1})^\beta = a (ba)^{-\beta-1}.$$

Case (ii): y and z w.c.r.

Since y and z are w.c.r., we have $\ell(y), \ell(z) \geq 3$. Let $x = x_1 \cdots x_p$, $y = y_1 \cdots y_n$, and $z = z_1 \cdots z_m$ in normal forms. Let $y' = y_1^{-1} y y_1 = y_2 \cdots (y_n y_1)$ and $z' = z_1^{-1} z z_1 = z_2 \cdots (z_m z_1)$. Then y' and z' are c.r. and $z_1^{-1} x^{-1} y_1 (y') y_1^{-1} x z_1 = z'$. Put $x' = y_1^{-1} x z_1$. Then $x'^{-1} y' x' = z'$, $\ell(y') \geq 2$, and by the previous case there exist a_1, b_1, α , and β such that $y' = (a_1 b_1)^\alpha$, $z' = (b_1 a_1)^\alpha$ and $x' = a_1 (b_1 a_1)^\beta$.

Case (1): $\ell(a_1) = 0$ or $\ell(b_1) = 0$.

Then $y' = C^\alpha$, $z' = (\delta^{-1} C \delta)^\alpha$ and $x' = C^s \delta$ for some $s \in \mathbb{Z}$, where $C = a_1 b_1$, and $\delta = a_1$ whenever $\ell(a_1) = 0$ and $\delta = b_1^{-1}$ whenever $\ell(b_1) = 0$.

Since y' is c.r., C is c.r. Thus we can write $C = C'(y_n y_1)$ in reduced form, for some C' . Now since $y_2 \cdots y_{n-1} (y_n y_1) = \delta z_2 \cdots z_{m-1} (z_m z_1) \delta^{-1}$, we have $y_n y_1 = \gamma z_m z_1 \delta^{-1}$ and that $y_2 \cdots y_{n-1} = \delta z_2 \cdots z_{m-1} \gamma^{-1}$ for some $\gamma \in B$.

Put $a = y_1 C' \gamma z_m$ and $b = z_1 \delta^{-1} y_1^{-1}$. Then

$$\begin{aligned} y &= y_1 C^{\alpha-1} C' y_n = y_1 (C' y_n y_1)^{\alpha-1} C' y_n = (y_1 C' y_n)^\alpha = \\ &= (y_1 C' \gamma z_m z_1 \delta^{-1} y_1^{-1})^\alpha = (ab)^\alpha. \end{aligned}$$

We also have

$$\begin{aligned} z &= z_1 (z_2 \cdots z_{m-1} z_m z_1) z_1^{-1} = z_1 \delta^{-1} C^\alpha \delta z_1^{-1} = z_1 (\delta^{-1} C' y_n y_1 \delta)^\alpha z_1^{-1} = \\ &= z_1 (\delta^{-1} C' \gamma z_m z_1)^\alpha z_1^{-1} = (z_1 \delta^{-1} C' \gamma z_m)^\alpha = (z_1 \delta^{-1} y_1^{-1} y_1 C' \gamma z_m)^\alpha = (ba)^\alpha. \end{aligned}$$

We see that $y = ab$ and $z = ba$ are in semi-reduced forms. We have

$$x = y_1 x' z_1^{-1} = y_1 C^s \delta z_1^{-1}.$$

If $s \geq 0$, then

$$\begin{aligned} x &= y_1 x' z_1^{-1} = y_1 (C' y_n y_1)^s \delta z_1^{-1} = (y_1 C' y_n)^s y_1 \delta z_1^{-1} = \\ &= (y_1 C' \gamma z_m z_1 \delta^{-1} y_1^{-1})^s y_1 \delta z_1^{-1} = (ab)^s b^{-1} = a (ba)^{s-1} \end{aligned}$$

If $s < 0$, then

$$\begin{aligned} x &= y_1 x' z_1^{-1} = y_1 (y_1^{-1} y_n^{-1} C'^{-1})^{-s} \delta z_1^{-1} = (y_n^{-1} C'^{-1} y_1^{-1})^{-s} y_1 \delta z_1^{-1} = \\ &= (y_1 C' \gamma z_m z_1 \delta^{-1} y_1^{-1})^s y_1 \delta z_1^{-1} = (ab)^s b^{-1} = a (ba)^{s-1} \end{aligned}$$

Case (2): $\ell(a_1) \neq 0$ and $\ell(b_1) \neq 0$.

Since $\ell(a_1) \neq 0$ and $\ell(b_1) \neq 0$, we can write $b_1 = B'(y_n y_1)$ and $a_1 = A'(z_m z_1)$ in reduced forms for some B' and A' . Put $a = y_1 A' z_m$ and $b = z_1 B' y_n$. Then

$$\begin{aligned} y &= y_1 y' y_1^{-1} = y_1 (a_1 b_1)^\alpha y_1^{-1} = y_1 (a_1 B' y_n y_1)^\alpha y_1^{-1} = \\ &= y_1 (a_1 B' y_n y_1)^{\alpha-1} a_1 B' y_n = (y_1 a_1 B' y_n)^\alpha = \\ &= (y_1 A' z_m z_1 B' y_n)^\alpha = (ab)^\alpha \end{aligned}$$

and

$$\begin{aligned} z &= z_1 z' z_1^{-1} = z_1 (b_1 a_1)^\alpha z_1^{-1} = z_1 (b_1 A' z_m z_1)^\alpha z_1^{-1} = \\ &= (z_1 b_1 A' z_m)^\alpha = (z_1 B' y_n y_1 A' z_m)^\alpha = (ba)^\alpha, \end{aligned}$$

and we see that ab and ba are in semi-reduced forms.

If $x' = a_1 (b_1 a_1)^\beta$ and $\beta \geq 0$, then

$$\begin{aligned} x &= y_1 x' z_1^{-1} = y_1 a_1 (b_1 a_1)^\beta z_1^{-1} = y_1 A' z_m z_1 (B' y_n y_1 A' z_m z_1)^\beta z_1^{-1} = \\ &= y_1 A' z_m (z_1 B' y_n y_1 A' z_m)^\beta = a(ba)^\beta. \end{aligned}$$

The case $x' = a_1 (b_1 a_1)^\beta$ and $\beta < 0$ can be treated similarly.

Case (iii): y w.c.r. and z c.r.

Let $x = x_1 \cdots x_p$, $y = y_1 \cdots y_n$, and $z = z_1 \cdots z_m$ in normal forms. Let $y' = y_1^{-1} y y_1 = y_2 \cdots (y_n y_1)$. Then y' is c.r. and $x^{-1} y_1 (y') y_1^{-1} x = z$. Put $x' = y_1^{-1} x$. Then $x'^{-1} y' x' = z$ and by case (i) there exist a_1 , b_1 , α , and β such that $y' = (a_1 b_1)^\alpha$, $z' = (b_1 a_1)^\alpha$ and $x' = a_1 (b_1 a_1)^\beta$. Then we consider the case $\ell(a_1) = 0$ or $\ell(b_1) = 0$, and the case $\ell(a_1) \neq 0$ and $\ell(b_1) \neq 0$. These two cases can be treated as the corresponding subcases (1-a) and (1-b) of case (i), taking care here of the fact that the corresponding elements a and b satisfy the following condition: ab is in reduced form and ba is in semi-reduced form.

This completes the proof of Theorem 6.1 in all cases. \square

7 Centralizer in valuated groups

This section is devoted to study some properties of centralizers in valuated groups with normal forms. The main subject is to show the following theorem.

Theorem 7.1 *Let (G, ℓ) be a valuated group with normal forms and let $g \in G$ be a c.r. element of length greater than 2. Then there exists a c.r. element s such that $C_G(g) = \langle s \rangle \times (B \cap C_G(g))$.*

The following lemma is a detailed version of [Hur81, Lemma 4.9].

Lemma 7.2 *Let (G, ℓ) be a valuated group with normal forms and let $g \in S \setminus B$. Then either*

- (1) $C_G(g) = C_G(h)^x$ for some $h, x \in G$, such that $h \in B$ and $x \in S \setminus B$, or
- (2) $C_G(g) \subseteq S \cap N$, or
- (3) $C_G(g) = \langle g \rangle \times (B \cap C_G(g))$ and $\ell(g^2) = 2$.

Proof

We suppose that (1) is not true and we show (2) or (3). We show the following claims.

Claim 1. If $a, b \in S \setminus B$ and $\ell(a^{-1}ba) = 1$, then $\ell(a^{-1}b) = 0$ or $\ell(ba) = 0$ or $\ell(a^{-1}b) = \ell(a^{-1}b) = 1$.

Proof. Suppose that $\ell(a^{-1}b) \neq 0$ and $\ell(ba) \neq 0$. We show that in that case we must have $\ell(a^{-1}b) = \ell(ab) = 1$. Indeed, if $\ell(a^{-1}b) = 2$ then by Proposition 3.5, the sequence (a^{-1}, ba) is S -reduced whenever $\ell(ba) = 1$, and the sequence (a^{-1}, b, a) is S -reduced whenever $\ell(ba) = 2$. But this contradicts $\ell(a^{-1}ba) = 1$. The situation is similar if we suppose that $\ell(ba) = 2$. \square

Claim 2. We have $C_G(g) = \langle S \cap C_G(g) \rangle$.

Proof. Let $x \in C_G(g)$ and $x = s_1 \cdots s_n$ in normal form. We prove by induction on n that $x \in \langle S \cap C_G(g) \rangle$. The conclusion is clear for $n = 1$. We have, for $n \geq 2$,

$$(1) \quad s_n^{-1} \cdots s_1^{-1} g s_1 \cdots s_n = g,$$

and thus $\ell(s_1^{-1} g s_1) \leq 1$. Since $g \notin B^x$ for every $x \in S \setminus B$, we get $\ell(s_1^{-1} g s_1) = 1$.

We claim that $\ell(s_1^{-1} g) = 0$ or $\ell(g s_1) = 0$. If it is not the case then, by Claim 1, $\ell(s_1^{-1} g) = \ell(g s_1) = 1$. But in that case, by the above proposition, since the sequence (s_1, s_2) is S -reduced and $\ell(s_1^{-1} g s_1) = 1$, the sequence $(s_1^{-1} g s_1, s_2)$ is S -reduced; similarly the sequence $(s_2^{-1}, s_1^{-1} g s_1)$ is S -reduced. Hence the sequence $(s_2^{-1}, s_1^{-1} g s_1, s_2)$ is S -reduced, and thus the sequence

$$(s_n^{-1}, \dots, s_2^{-1}, s_1^{-1} g s_1, s_2, \dots, s_n)$$

is S -reduced; a contradiction with $\ell(g) = 1$.

Thus $\ell(s_1^{-1} g) = 0$ or $\ell(g s_1) = 0$ as claimed. Hence $g = s_1 h$ or $g = h s_1^{-1}$ for some $h \in B$. Thus, replacing in (1), we have

$$s_n^{-1} \cdots s_2^{-1} h s_1 \cdots s_n = g, \text{ when } g = s_1 h,$$

$$s_n^{-1} \cdots s_2^{-1} s_1^{-1} h s_2 \cdots s_n = g, \text{ when } g = h s_1^{-1},$$

which can be rewritten as

$$s_n^{-1} \cdots s_2^{-1} h g h^{-1} s_2 \cdots s_n = g, \text{ when } g = s_1 h,$$

$$s_n^{-1} \cdots s_2^{-1} h^{-1} g h s_2 \cdots s_n = g, \text{ when } g = h s_1^{-1}.$$

By induction,

$$h^{-1} s_2 \cdots s_n \in \langle S \cap C_G(g) \rangle, \text{ or } h s_2 \cdots s_n \in \langle S \cap C_G(g) \rangle,$$

depending on the case $g = s_1 h$ or $g = h s_1^{-1}$. Therefore

$$s_1 \cdots s_n = (s_1 h) h^{-1} s_2 \cdots s_n \in \langle S \cap C_G(g) \rangle, \text{ when } g = s_1 h,$$

$$s_1 \cdots s_n = (s_1 h^{-1}) h s_2 \cdots s_n \in \langle S \cap C_G(g) \rangle, \text{ when } g = h s_1^{-1},$$

and this completes the proof of the claim. \square

Claim 3.

- (i) If $\ell(g^2) \leq 1$ then $C_G(g) \subseteq S \cap N$.
- (ii) If $\ell(g^2) = 2$ then $C_G(g) = \langle g \rangle \times (B \cap C_G(g))$.

Proof.

(i) By Claim 2, it is sufficient to show that if $s_1, s_2 \in S \cap C_G(g)$ then $s_1 s_2 \in S \cap N$.

Let us show first that if $s \in S \cap C_G(g)$ then $s \in N$. Since $\ell(s^{-1}gs) = 1$, by Claim 1, we have $\ell(s^{-1}g) \leq 1$ and $\ell(gs) \leq 1$. Therefore

$$c(g, s^{-1}) \geq \frac{1}{2}, c(g^{-1}, s) \geq \frac{1}{2}.$$

Since $\ell(g^2) \leq 1$ we get $c(g, g^{-1}) \geq \frac{1}{2}$. Using axiom \mathcal{A}_3 we find that $c(s, s^{-1}) \geq \frac{1}{2}$ and thus $\ell(s^2) \leq 1$ as desired.

By Claim 2, it is sufficient to show that if $s_1, s_2 \in S \cap C_G(g)$ then $s_1 s_2 \in S \cap N$. Let us show now that $s_1 s_2 \in S$. Suppose that $\ell(s_1 s_2) = 2$; in particular we have $\ell(s_1) = \ell(s_2) = 1$. Since $\ell(s_1^{-1}gs_1) = 1$, by Claim 1, we get $\ell(s_1^{-1}g) = 0$ or $\ell(gs_1) = 0$. Therefore $gs_2 = h s_1 s_2$ whenever $\ell(s_1^{-1}g) = 0$ and $s_2^{-1}g = h s_2^{-1} s_1^{-1}$ whenever $\ell(s_1^{-1}g) = 0$, for some $h \in B$. Hence $\ell(gs_2) = 2$ or $\ell(s_2^{-1}g) = 2$. Since $\ell(s_2^{-1}gs_2) = 1$, again by Claim 1, $\ell(s_2^{-1}g) = 0$ whenever $\ell(gs_2) = 2$; and $\ell(gs_2) = 0$ whenever $\ell(s_2^{-1}g) = 2$. We conclude that

$$(2) \quad c(g^{-1}, s_2) \geq \frac{1}{2} \text{ and } c(g, s_2^{-1}) = 0, \text{ or}$$

$$(3) \quad c(g^{-1}, s_2) = 0 \text{ and } c(g, s_2^{-1}) \geq \frac{1}{2}.$$

Since $c(g, g^{-1}) \geq \frac{1}{2}$ and $c(s_2, s_2^{-1}) \geq \frac{1}{2}$ we find, using axiom \mathcal{A}_3 ,

$$c(g, s_2^{-1}) \geq \frac{1}{2} \text{ if } c(g^{-1}, s_2) \geq \frac{1}{2}, \text{ and}$$

$$c(g^{-1}, s_2) \geq \frac{1}{2} \text{ if } c(g, s_2^{-1}) \geq \frac{1}{2},$$

a contradiction with (2) and (3). Therefore $s_1 s_2 \in S$ as desired.

(ii) By Claim 2, it is sufficient to show that if $s \in S \cap C_G(g)$ then $s \in \langle g, C_G(g) \cap B \rangle$ and $\langle g \rangle \cap C_G(g) \cap B = 1$.

Let $s \in S \cap C_G(g)$. We claim that $\ell(s^{-1}g) = 0$ or $\ell(gs) = 0$. If it is not the case then, by Claim 1, $\ell(s^{-1}g) = \ell(gs) = 1$. But in that case, we have

$$c(g^{-1}, s) \geq \frac{1}{2} \text{ and } c(g, s^{-1}) \geq \frac{1}{2},$$

and by using axiom \mathcal{A}_3 we find, since $c(g, g^{-1}) = 0$, $c(s, s^{-1}) = 0$. Therefore $\ell(s^2) = 2$ and thus the sequence (s, s) is S -reduced. By the above proposition, since $\ell(s^{-1}gs) = 1$, the sequence $(s^{-1}gs, s)$ is S -reduced; similarly the sequence $(s^{-1}, s^{-1}gs)$ is S -reduced. Hence the sequence $(s^{-1}, s^{-1}gs, s)$ is S -reduced; a contradiction with $\ell(g) = 1$.

Hence $\ell(s^{-1}g) = 0$ or $\ell(gs) = 0$ as claimed and thus $s \in \langle g, C_G(g) \cap B \rangle$. Now the fact that $\langle g \rangle \cap C_G(g) \cap B = 1$ follows from the fact that $\ell(g^p) = |p|$ for any $p \in \mathbb{Z}$. This ends the proof of the claim and of the lemma. \square

The following lemma can be found in [Hur81, Lemma 4.9(ii)]. We give a new proof of it by using Theorem 6.1.

Lemma 7.3 *Let (G, ℓ) be a valuated group with normal forms. Let $x, y \in G$ such that x satisfies $\ell(x^2) = 2\ell(x)$, and $[x, y] = 1$. Then there exist X in G , and h_1, h_2 in B , and $m, n \in \mathbb{Z}$, such that:*

$$x = h_1 X^n, \quad y = h_2 X^m, \quad [h_1, X] = [h_2, X] = [h_1, h_2] = 1,$$

and if $\ell(x) \geq 1, \ell(y) \geq 1$ then $\ell(y^2) = 2\ell(y)$.

Proof

We prove the lemma by induction on $\ell(x) = p$.

For $p = 0$. By taking $h_1 = x, h_2 = 1, X = y, n = 0, m = 1$ we find the desired conclusion.

For $p = 1$. Since $\ell(x^2) = 2\ell(x)$, by Lemma 7.2, we find

$$C_G(x) = \langle x \rangle \times (B \cap C_G(x))$$

and the conclusion is clear.

We go from p to $p + 1$. By Theorem 6.1, there exist a, b in G , and n and m in \mathbb{Z} , with $n \geq 1$, such that $x = (ab)^n = (ba)^n$, $y = a(ba)^m$ and ab and ba are in reduced forms.

We claim that $\ell(y^2) = 2\ell(y)$. Suppose first that $\ell(a) = 0$ or $\ell(b) = 0$. Then, we get, $x = C^n = \delta^{-1}C^n\delta$ and $y = C^s\delta$ for some $s \in \mathbb{Z}$, where $C = ab$, and $\delta = a$ whenever $\ell(a) = 0$ and $\delta = b^{-1}$ whenever $\ell(b) = 0$.

Since x is c.r., C is c.r. We have

$$\ell(x^2) = \ell(\delta x^2) = \ell(x\delta x) = \ell(C^m \delta C^m),$$

and by using Lemma, we conclude $\ell(C\delta C) = 2\ell(C)$, and thus

$$\ell(y^2) = \ell(C^s \delta C^s) = 2|s|\ell(C) = 2\ell(y),$$

as claimed. Now we suppose that $\ell(a) \neq 0$ and $\ell(b) \neq 0$. Since x is c.r. and $(ab)^n = (ba)^n$, it follows that a and b are c.r. Therefore

$$\ell(y^2) = \ell(a(ba)^m a(ba)^m) = \ell(a) + m\ell(ba) + \ell(a) + m\ell(ba) = 2\ell(y),$$

whenever $m \geq 0$, and

$$\ell(y^2) = \ell(a(a^{-1}b^{-1})^{-m}a(a^{-1}b^{-1})^{-m}) = \ell(((b^{-1}a^{-1})^{-m-1}b^{-1})^2) = 2\ell(y),$$

whenever $m \leq 0$. This ends the proof of the claim.

Let $0 \leq t \leq n-1$ in \mathbb{N} and $k \in \mathbb{Z}$, such that $m = kn + t$. Let $z = a(ba)^t$. Suppose first that $t = n-1$ and $\ell(b) = 0$. Then

$$y = a(ba)^m = a(ba)^t(ba)^{nk} = (ab)^nb^{-1}((ba)^n)^k = xb^{-1}x^k,$$

and since $[b, x] = 1$ we find $y = x^{k+1}b^{-1}$. Therefore by taking $h_1 = 1, h_2 = b^{-1}, X = x$ we find the desired conclusion.

Now we suppose that $0 \leq t < n-1$ or $\ell(b) \neq 0$. Then, as above, z is c.r. and

$$\ell(z) = \ell(a(ba)^t) \leq \ell(a) + t\ell(ba) < \ell(x) = n.\ell(ba).$$

Since z is c.r. and $[z, x] = 1$, by induction, there exist X in G , and h_1, h_2 in B , such that:

$$z = a(ba)^t = h_1X^r, \text{ for some } r \in \mathbb{Z},$$

$$x = (ba)^n = h_2X^s, \text{ for some } s \in \mathbb{Z},$$

$$[h_1, X] = [h_2, X] = [h_1, h_2] = 1.$$

Thus we have:

$$x = (ba)^n = h_2X^s$$

$$y = a(ba)^t \cdot ((ba)^n)^k = h_1X^r(h_2X^s)^k = h_1h_2^kX^{r+sk},$$

and we find the desired conclusion. This ends the proof of the lemma. \square

The following is an immediate consequence of the precedent lemma.

Corollary 7.4 *Let (G, ℓ) be a valuated group with normal forms and $g \in G$ be a c.r. element such that $\ell(g) \geq 2$. Let $x \in C_G(g) \setminus B$. Then x is c.r. and either*

(1) $x = a^n, g = a^m$ for some $n, m \in \mathbb{Z}^\#$ and for some $a \in S \setminus B$ such that $\ell(a^2) = 2$ or,

(2) $\ell(x) \geq 2$. \square

Now we are ready to prove Theorem 7.1.

Proof of Theorem 7.1. Let $C = C_G(g)$ equipped with the induced length function. Then (C, ℓ) is a valuated group. We claim that (C, ℓ) satisfies the axiom \mathcal{A}_0^* of Theorem 5.1. If $x \in C$, then by Corollary 7.4 either $x \in B \cap C$ or x is c.r. and hence $\ell(x^2) > \ell(x)$. Therefore, if $x \notin B' = B \cap C$, then $\ell(x^2) > \ell(x)$; thus (C, ℓ) satisfies \mathcal{A}_0^* as claimed. Therefore by Theorem 5.1, $C = *_B B'(t_i, f_i)$, where $B'(t_i, f_i) = \langle B', t_i | f(x) = x^{t_i} \rangle$. But since the center of C contains a c.r. element, we get that $C = \langle B', t_i | f(x) = x^{t_i} \rangle$, which can be written simply as $C = \langle B', s | B'^s = B' \rangle$. Again, since $Z(C)$ contains a c.r. element, we find that $C = \langle s \rangle \times B'$, as desired. \square

We end this section with the following theorem. We give a proof of it, using only Corollary 7.4 and Theorem 4.1.

Theorem 7.5 *Let (G, ℓ) be a valuated group with normal forms. Let $g \in G$ be a c.r. element of length greater than 2. Then the following properties are equivalent:*

- (1) $C_G(g) \cap B = 1$.
- (2) $C_G(g)$ is infinite cyclic.
- (3) $C_G(g)$ is locally cyclic.

In that case, if G has no involutions then $C_G(g)$ is selfnormalizing.

Proof

- (1) \Rightarrow (2) By Corollary 7.4, for every $x \in C_G(g)$, $x \neq 1$, x is c.r. and we have:
- (i) $x = a^n$, $g = a^m$ for some $n, m \in \mathbb{Z}^\#$ and for some $a \in S \setminus B$ such that $\ell(a^2) = 2$ or,
 - (ii) $\ell(x) \geq 2$.

We claim that $C_G(g) \cap B^y = 1$, for every $y \in G$. Suppose towards a contradiction that $C_G(g) \cap B^y \neq 1$ for some $y \in G$ and let $z \in C_G(g) \cap B^y$. Then by the above property z is c.r. and $\ell(z^2) > \ell(z)$, a contradiction.

Hence by Theorem 4.1, $C_G(g)$ is a free product. The result follows.

(2) \Rightarrow (3) Obvious.

(3) \Rightarrow (2) We claim that $C_G(g) \cap B^y = 1$, for every $y \in G$. Suppose towards a contradiction that $C_G(g) \cap B^y \neq 1$ for some $y \in G$ and let $z \in C_G(g) \cap B^y$. Then $y = b^n$ and $g = b^m$ for some $n, m \in \mathbb{Z}^\#$ and for some b . Now since g is c.r., b is c.r. Thus z is c.r., a contradiction. Hence by Theorem 4.1, $C_G(g)$ is a free product. The result follows.

(2) \Rightarrow (1) Obvious.

Let us show now that $C_G(g)$ is selfnormalizing. Let $C_G(g) = \langle s \rangle$, for some $s \in G$. Then we see that s is c.r. and is not a proper power.

Let $x \in N(C_G(g))^\#$, then $s^x = s^m$ for some $m \in \mathbb{Z}^\#$. Now since s is not a proper power and c.r. we have $m = \pm 1$. If $m = 1$ we have the result. If $m = -1$ then $x^{-2}sx^2 = s$ and thus $x^2 \in C_G(g)$. Now since G has no involution, $x^2 \neq 1$. And since $C_G(g) \cap B = 1$, $\ell(x^2) \neq 0$, and by Lemma 7.4, x^2 is c.r. Hence x is c.r. and $\ell(x^2) > \ell(x)$.

Hence, as above, $N(C_G(g)) \cap B^y = 1$, for every $y \in G$.

Thus by Theorem 4.1, $A = N(C_G(g))$ is a free group. But since $N_A(C_G(g)) = A$ we have A is infinite cyclic. Hence A is generated by s since s is not a proper power. \square

8 The CSA property in valuated groups

If G is a group, S a subset of G and H is a subgroup, we say that H is *S-malnormal* if $H \cap H^s = 1$ for any $s \in S$, $s \neq 1$. The subject of this section is to prove the following theorem.

Theorem 8.1 *Let (G, ℓ) be a valuated group with normal forms and without involutions. Then the following properties are equivalent:*

- (1) G is a CSA^* -group.

(2) *The following properties are satisfied:*

- (i) *for every $g \in G$, $g \neq 1$, if $C_G(g) \subseteq S$ then $C_G(g)$ is abelian and S -malnormal,*
- (ii) *for every $g \in B$, $g \neq 1$, $C_G(g)$ is abelian and malnormal.*

Proof

Obviously we have (1) \Rightarrow (2). Assume (2) and let us show (1). We are going to prove that for every $g \in G^\#$, $C_G(g)$ is abelian and selfnormalizing. Let us treat first the case when $\ell(g) \leq 1$. The case $\ell(g) = 0$ follows from the assumption (ii). Suppose that $\ell(g) = 1$. By Lemma 7.2, there is three cases to consider:

- (a) $C_G(g) = C_G(h)^x$ for some $h, x \in G$, such that $\ell(h) = 0$ and $\ell(x) = 1$, or
- (b) $C_G(g) \subseteq S$, or
- (c) $C_G(g) = \langle g \rangle \times (B \cap C_G(g))$ and $\ell(g^2) = 2$.

The case (a) follows from the assumption (ii). Let us treat the case (b). By assumption (i), $C_G(g)$ is abelian and thus we prove that it is malnormal. We suppose also that $g \notin B^x$ for every $x \in G$ such that $\ell(x) = 1$, for if we are in the case (a).

Let $x \in G$ and $g', g'' \in C_G(g)^\#$ such that $g'^x = g''$. If $x \in S$, then $x \in C_G(g)$ because $C_G(g)$ is S -malnormal. Suppose now that $x \notin S$, i.e. that $x = s_1 \cdots s_n$ is in normal form with $n \geq 2$. Then

$$s_n^{-1} \cdots s_1^{-1} g' s_1 \cdots s_n g''^{-1} = 1.$$

This implies that $\ell(s_1^{-1} g' s_1) = 0$, or $\ell(s_1^{-1} g' s_1) = 1$.

If $\ell(s_1^{-1} g' s_1) = 0$ then $g' \in C_G(\gamma)^{-s_1}$ for some $\gamma \in B$. Now since $g' \in C_G(\gamma)^{-s_1}$ and the last group is abelian and malnormal, $g \in C_G(\gamma)^{-s_1}$, thus $C_G(g) \subseteq C_G(\gamma)^{-s_1}$. Since $C_G(\gamma)^{-s_1}$ is malnormal, $x \in C_G(\gamma)^{-s_1}$. Finally $x \in C_G(\gamma)^{-s_1}$, as the last group is abelian.

We claim that the case $\ell(s_1^{-1} g' s_1) = 1$ cannot occur. For if $\ell(s_1^{-1} g' s_1) = 1$, then $g' = s_1$ and $\ell(s_2 g' s_2^{-1}) = 1$. (If $\ell(s_2 g' s_2^{-1}) = 0$ then we are in the previous case). If $n = 2$, then $s_2 g' s_2^{-1} = g''$ and $s_2 \in C_G(g)$ and $\ell(g' s_1) = 2$, a contradiction. If $n > 2$ then $g' = s_2$ and $\ell(g'^2) = 2$, which is also a contradiction as $g'^2 \in C_G(g)$. This completes the proof of the case (b).

We treat now the case (c).

If $B \cap C_G(g) \neq 1$, then let $h \in (B \cap C_G(g))^\#$. Then $C_G(g) \subseteq C_G(h)$, as $C_G(h)$ is abelian and malnormal. We see that also in this case that $C_G(g)$ is malnormal.

If $B \cap C_G(g) = 1$, then $C_G(g) = \langle g \rangle$, and by Theorem 7.5, $C_G(g)$ is selfnormalizing.

Let us now treat the case when $\ell(g) \geq 2$. Then $g = s g' s^{-1}$, where g' is c.r. Now if $\ell(g') \leq 1$ then, up to conjugacy, we are in the previous case. Thus suppose that $\ell(g') \geq 2$. Then we see as before that if $B \cap C_G(g) \neq 1$, then $C_G(g)$ is abelian and selfnormalizing, and if $B \cap C_G(g) = 1$, then by Theorem 7.5, $C_G(g)$ is infinite cyclic and selfnormalizing. \square

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